

# A remark on recent lower bounds for nodal sets

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## Abstract

Recently, two papers ([SZ, CM]) appeared which give lower bounds on the size of the nodal sets of eigenfunctions. The purpose of this short note is to point out a third method to obtain a power law lower bound on the volume of the nodal sets. Our method is based on the Donnelly-Fefferman growth bound for eigenfunctions and a growth vs. volume relation we proved in [M].

## 1 Introduction and Background

Consider a  $C^\infty$  Riemannian manifold  $(M, g)$ . Let  $\Delta$  be the Laplace-Beltrami operator on  $M$ . The eigenfunctions are solutions of  $\Delta\varphi_\lambda + \lambda\varphi_\lambda = 0$ . We are interested in finding lower bounds on the size of the nodal set in the case where  $(M, g)$  is  $C^\infty$  but not real analytic.

Yau's Conjecture ([Y]) asserts that the size of the nodal set is comparable to  $\lambda^{1/2}$ . Donnelly and Fefferman proved ([DF]) Yau's conjecture in case  $(M, g)$  is real analytic. The real analyticity assumption is used in a crucial way: The eigenfunctions are analytically continued to holomorphic functions with bounded growth, and then the problem is reduced to a problem about polynomials.

The history of the lower bounds in the  $C^\infty$  but non real-analytic case can be summarized as follows: In dimension two the lower bound in Yau's conjecture was proved by Brüning in [B], and by Yau, independently. In dimension  $n = 3$  it is known that the size of the nodal set is bounded away from 0 by a constant independent of  $\lambda$ , due to the recent work of Colding and Minicozzi ([CM]). In dimensions  $n \geq 4$  all known lower bounds today are *decreasing* to 0 with  $\lambda$ . In fact, from [DF] and [HL] they were known to be exponentially decreasing. The recent developments by Sogge-Zelditch ([SZ]) and Colding-Minicozzi ([CM]) give polynomially decreasing bounds. In this note (Theorem 2.3) we extract polynomially decreasing bounds in a few lines from our previous work in [M].

### 1.1 Background - the work of Donnelly and Fefferman

We recall three of the many innovative ideas proved in [DF], which frequently appear in the next sections.

- I. Let  $B \subset M$  be a metric ball.  $\frac{1}{2}B$  is a concentric ball half the radius of  $B$ . Define the growth of  $\varphi_\lambda$  in  $B$  by

$$\beta(\varphi_\lambda; B) = \log \frac{\max_B |\varphi_\lambda|}{\max_{\frac{1}{2}B} |\varphi_\lambda|}.$$

Then, for every ball  $B$

$$\beta(\varphi_\lambda; B) \leq C_{(M,g)} \lambda^{1/2}. \quad (1.1)$$

This is true for any  $C^\infty$ -manifold.

- II. In the *real analytic* case, for each eigenvalue  $\lambda$  one can find disjoint balls of radius  $c\lambda^{-1/2}$ , the total volume of which is at least  $C\text{Vol}(M)$ , and such that the growth of the eigenfunction in each of these balls is at most  $\beta_0$ , where  $\beta_0$  is a constant *independent of  $\lambda$* , and in addition the eigenfunction vanishes at the center of each such ball.
- III. There exists a relation between growth estimates and volume estimates: In each ball in which the growth of the eigenfunction is at most  $\beta_0$ , and in which the eigenfunction vanishes at a point of its middle half the volume of the positive set, the volume of the negative set, and the volume of the ball are all comparable to each other. This relation is true in the general  $C^\infty$ -case.

From II and III one obtains lower bounds on the size of the set  $\{\varphi_\lambda = 0\} \cap B$  by the relative isoperimetric inequality ([F]): Let  $A_1, A_2 \subset B$  be open subsets. Then

$$\text{Vol}_{n-1}(\partial A_1 \cap \partial A_2) \geq C \min\{(\text{Vol}_n A_1)^{\frac{n-1}{n}}, (\text{Vol}_n A_2)^{\frac{n-1}{n}}\}, \quad (1.2)$$

where  $\text{Vol}_{n-1}$  is the Hausdorff measure. In our situation  $A_1 = \{\varphi_\lambda > 0\} \cap B$ ,  $A_2 = \{\varphi_\lambda < 0\} \cap B$ . We get that the  $(n-1)$ -volume of the set  $\{\varphi_\lambda = 0\}$  in each ball of the collection in II is comparable to the ball's boundary area. Finally, multiplying this estimate by the number of balls in the collection ( $c\lambda^{n/2}$ ) gives a lower bound of  $c\lambda^{1/2}$ .

## 2 An estimate using the Growth Bound of Donnelly and Fefferman

Our approach to lower bounds in the  $C^\infty$  case is to give an estimate on the positivity volume in *every* ball for which the eigenfunction vanishes at its middle half. In this way we circumvent the need to estimate the number of balls in which the eigenfunction has bounded growth (cf. idea II. in Section 1.1).

In [M] we have shown that in every ball  $B$  for which  $\varphi_\lambda$  vanishes at  $\frac{1}{2}B$  one has

$$\frac{\text{Vol}(\{\varphi_\lambda > 0\} \cap B)}{\text{Vol} B} \geq C\beta(\varphi_\lambda; B)^{-(n-1)}. \quad (2.1)$$

Symmetrically, the same estimate is true also for the negativity set. The proof of (2.1) is based on an iteration procedure which starts with an exponentially small lower bound. To explain the basic idea, we let  $u$  be a harmonic function in the unit ball. Suppose for simplicity  $u(0) > 0$ . We normalize  $u$  so that  $u(0) = 1$ . Suppose  $u < M = e^\beta$  in  $B_1$ . Then the mean value property immediately gives that  $\text{Vol}(\{u > 0\}) > C_1 M^{-1}$ . Now we improve this primary estimate by iteration: Consider the ball  $B_{1/2}$ . If  $u \leq M^{1/2}$  on  $B_{1/2}$ , then the same argument as above gives  $\text{Vol}(\{u > 0\}) > C_2 M^{-1/2}$ . Otherwise, there exists a point  $x$  such that  $|x| = 1/2$ ,  $u(x) > M^{1/2}$ . Consider the ball  $B = B(x, 1/2)$ . Since  $(\sup_B u)/u(x) < M^{1/2}$ , applying the above argument to the ball  $B(x, 1/2)$  gives again  $\text{Vol}(\{u > 0\}) < C_3 M^{-1/2}$ . Thus, in any case  $\text{Vol}(\{u > 0\}) < C_4 M^{-1/2}$ . We can continue this sequence of improvements to obtain  $\text{Vol}(\{u > 0\}) < C_\varepsilon M^{-\varepsilon}$  for all  $\varepsilon > 0$ . Optimizing, one gets in this way the bound  $C(\log M)^{-n} = C\beta^{-n}$ . A slight modification of this argument (see [M]) gives  $C\beta^{-(n-1)}$ . The case where  $u(0) = 0$  is a little more involved, since we have to take into consideration the different signs of  $u$ . We overcome this difficulty by applying the Harnack inequality. Finally, it turns out that the proof for harmonic functions can be adapted to solutions of second order  $C^\infty$  elliptic equations.

Plugging the estimate (2.1) for the positivity set, the same estimate for the negativity set and (1.1) in (1.2) we obtain

$$\frac{\text{Vol}_{n-1}(\{\varphi_\lambda = 0\} \cap B)}{\text{Vol}_{n-1}(\partial B)} \geq C\lambda^{-\frac{(n-1)^2}{2n}} \quad (2.2)$$

Finally, it is well known (and an easy fact) that for each  $\lambda$  one can find a set of disjoint balls of radius  $c\lambda^{-1/2}$  such that the eigenfunction vanishes at the middle half of each such ball, and the total volume of which is at least  $C\text{Vol}(M)$ . Hence, one multiplies the estimate (2.2) by the number of such balls ( $C\lambda^{n/2}$ ) and obtains

**Theorem 2.3.**

$$\text{Vol}_{n-1}(\varphi_\lambda = 0) \geq C\lambda^{-\frac{(n-1)^2}{2n}} \lambda^{\frac{1-n}{2}} \lambda^{n/2} = C\lambda^{\frac{3-n}{2} - \frac{1}{2n}}.$$

## 3 The idea of Colding and Minicozzi

Colding and Minicozzi give in [CM] a new argument that shows that on any  $C^\infty$ -Riemannian manifold one can find a constant  $\beta_0$  and for each eigenvalue  $\lambda$  a disjoint set of balls of radius  $c\lambda^{-1/2}$  such that the growth of  $\varphi_\lambda$  in each such ball is bounded by  $\beta_0$  and such that the *total*  $L^2$ -norm of  $\varphi_\lambda$  on the union of these balls  $G$  is at least  $\frac{3}{4}\|\varphi_\lambda\|_{L^2(M)}$ , and in addition the eigenfunction vanishes at the center of each such ball. This should be compared with idea II in Section 1.1.

Now, one would like to estimate the *number* of balls in  $G$ . Since the  $L^2$ -norm of  $\varphi_\lambda$  on  $G$  is big, we can apply Hölder's inequality and upper  $L^p$ -bounds for  $p > 2$ , in order to obtain a lower bound on the *volume* of  $G$ . The easiest such bounds are the Sobolev bounds:

$$\|\varphi_\lambda\|_{L^p} \leq \lambda^{\frac{n}{2}(\frac{1}{2}-\frac{1}{p})} \|\varphi_\lambda\|_{L^2} .$$

The sharp  $L^p$  bounds are Sogge estimates ([S, Ch. 5]) (which in the  $p = \infty$  case reduce to the bound coming from local Weyl law):

$$\|\varphi_\lambda\|_{L^p(M)} \leq \lambda^{\delta(p)} \|\varphi_\lambda\|_{L^2(M)} ,$$

where

$$\delta(p) = \begin{cases} \frac{n-1}{4}(\frac{1}{2} - \frac{1}{p}), & 2 \leq p \leq \frac{2(n+1)}{n-1} \\ \frac{n}{2}(\frac{1}{2} - \frac{1}{p}) - \frac{1}{4}, & \frac{2(n+1)}{n-1} \leq p \leq \infty \end{cases}$$

If we take  $p = 2(n+1)/(n-1)$ , we get the following lower bound on the volume of  $G$ :

$$\text{Vol}(G) > C\lambda^{-(n-1)/4} .$$

Hence, the number of balls in  $G$  is at least  $\lambda^{(n+1)/4}$ . Then we proceed as before to get

$$\text{Vol}_{n-1}(\{\varphi_\lambda = 0\}) \geq C\lambda^{(n+1)/4+(1-n)/2} = \lambda^{(3-n)/4} .$$

## 4 The method of Sogge-Zelditch

Sogge and Zelditch were inspired in [SZ] by Dong's formula ([D]). In particular, they prove:

$$\lambda \int_M |\varphi_\lambda| d\text{Vol} = 2 \int_{\{\varphi_\lambda=0\}} |\nabla \varphi_\lambda| d\text{Area} . \quad (4.1)$$

To the preceding formula one can join upper pointwise bounds on  $\nabla \varphi_\lambda$  coming from the local Weyl formula:

$$|\nabla \varphi_\lambda| \leq \lambda^{(n+1)/4} . \quad (4.2)$$

Sogge's  $L^p$ -upper bounds on  $\varphi_\lambda$  also give lower  $L^1$ -bounds. Indeed, by Hölder's inequality:

$$1 = \|\varphi_\lambda\|_{L^2}^2 \leq \|\varphi_\lambda\|_{L^1}^{\frac{p-2}{p-1}} \|\varphi_\lambda\|_{L^p}^{\frac{p}{p-1}} . \quad (4.3)$$

Thus,

$$\|\varphi_\lambda\|_{L^1} \geq \|\varphi_\lambda\|_{L^p}^{-\frac{p}{p-2}} \geq \lambda^{-\frac{p\delta(p)}{p-2}} .$$

If we choose  $p = 2(n+1)/(n-1)$  we obtain

$$\|\varphi_\lambda\|_{L^1} \geq C\lambda^{-(n-1)/8} . \quad (4.4)$$

From (4.2), (4.4) and (4.1) one obtains

$$\lambda^{(n+1)/4} \text{Vol}_{n-1}(\{\varphi_\lambda = 0\}) \geq C\lambda \cdot \lambda^{(1-n)/8} ,$$

and after rearranging

$$\text{Vol}_{n-1}(\{\varphi_\lambda = 0\}) \geq C\lambda^{(7-3n)/8} .$$

## 5 Conclusion

We conclude by a short summary of the three methods discussed above:

The idea in [CM] is closest in spirit to the work of [DF]: The number of disjoint balls of the wavelength radius centered on the nodal set and in which the growth of the eigenfunction is bounded is estimated using Sogge's estimates. Since the size of the nodal set in each such ball is comparable to the size of the boundary of the ball, a lower bound on the size of the nodal set is obtained. This method gives the best known bounds today.

Our approach from [M] gives an estimate of the size of the nodal set in *any* ball in terms of the growth of the eigenfunction in the ball. It uses inequality (1.1) to bound the growth in the worst case. In particular, we circumvent the estimate of the number of balls with bounded growth. Our estimates are not sharp. Hence, it seems that room for strengthening the result is still left.

The method of [SZ] is based on expressing the  $L^1$ -norm of the eigenfunction as an integral of the gradient over the nodal set. This is close in spirit to [D]. Pointwise gradient estimates from the local Weyl law and a sharp  $L^1$ -lower bound are applied.

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